Exercise 8

Use a residue and the contour shown in Fig. 95, where $R > 1$, to establish the integration formula

$$\int_0^\infty \frac{dx}{x^3+1} = \frac{2\pi}{3\sqrt{3}}.$$

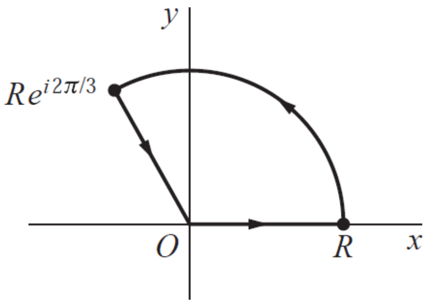


FIGURE 95

Solution

In order to evaluate the integral, consider the corresponding function in the complex plane,

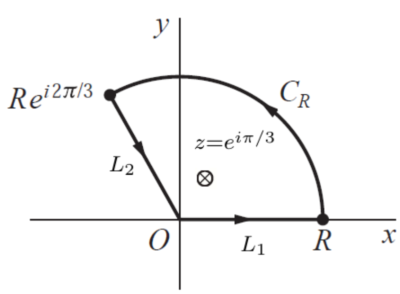
$$f(z) = \frac{1}{z^3+1},$$

and the contour in Fig. 95. Singularities occur where the denominator is equal to zero.

$$z^3 + 1 = 0$$

$$z = \sqrt[3]{1}\exp\left[i\left(\frac{\pi+2k\pi}{3}\right)\right], \quad k = 0,1,2 \quad \rightarrow \quad \begin{cases} z_1 = e^{i\pi/3} = \frac{1}{2} + i\frac{\sqrt{3}}{2} \\ z_2 = e^{i\pi} = -1 \\ z_3 = e^{i5\pi/3} = \frac{1}{2} - i\frac{\sqrt{3}}{2} \end{cases}$$

The singular point of interest to us is the one that lies within the closed contour, $z = e^{i\pi/3}$.



According to Cauchy's residue theorem, the integral of $1/(z^3+1)$ around the closed contour is equal to $2\pi i$ times the sum of the residues at the enclosed singularities.

$$\oint_C \frac{dz}{z^3+1} = 2\pi i \operatorname*{Res}_{z=z_1} \frac{1}{z^3+1}$$

This closed loop integral is the sum of three integrals, one over each arc in the loop.

$$\int_{L_1} \frac{dz}{z^3+1} + \int_{L_2} \frac{dz}{z^3+1} + \int_{C_R} \frac{dz}{z^3+1} = 2\pi i \operatorname*{Res}_{z=z_1} \frac{1}{z^3+1}$$

The parameterizations for the arcs are as follows.

$$\begin{aligned} L_1 :& \quad z = re^{i0}, && r = 0 \quad \to \quad r = R \\ L_2 :& \quad z = re^{i2\pi/3}, && r = R \quad \to \quad r = 0 \\ C_R :& \quad z = Re^{i\theta}, && \theta = 0 \quad \to \quad \theta = \frac{2\pi}{3} \end{aligned}$$

As a result,

$$\begin{aligned} 2\pi i \operatorname*{Res}_{z=z_1} \frac{1}{z^3+1} &= \int_0^R \frac{dr\, e^{i0}}{(re^{i0})^3+1} + \int_R^0 \frac{dr\, e^{i2\pi/3}}{(re^{i2\pi/3})^3+1} + \int_{C_R} \frac{dz}{z^3+1} \\ &= \int_0^R \frac{dr}{r^3+1} + \int_R^0 \frac{dr\, e^{i2\pi/3}}{r^3e^{i2\pi}+1} + \int_{C_R} \frac{dz}{z^3+1} \\ &= \int_0^R \frac{dr}{r^3+1} - \int_0^R \frac{dr\, e^{i2\pi/3}}{r^3+1} + \int_{C_R} \frac{dz}{z^3+1} \\ &= (1 - e^{i2\pi/3}) \int_0^R \frac{dr}{r^3+1} + \int_{C_R} \frac{dz}{z^3+1}. \end{aligned}$$

Take the limit now as $R \to \infty$. The integral over C_R consequently tends to zero. Proof for this statement will be given at the end.

$$(1 - e^{i2\pi/3}) \int_0^\infty \frac{dr}{r^3+1} = 2\pi i \operatorname*{Res}_{z=z_1} \frac{1}{z^3+1}$$

The denominator can be written as $z^3 + 1 = (z - z_1)(z - z_2)(z - z_3)$. From this we see that the multiplicity of the $z - z_1$ factor is 1. The residue at $z = z_1$ can then be calculated by

$$\operatorname*{Res}_{z=z_1} \frac{1}{z^3+1} = \phi(z_1),$$

where $\phi(z)$ is equal to $f(z)$ without the $z - z_1$ factor.

$$\phi(z) = \frac{1}{(z-z_2)(z-z_3)} \quad \Rightarrow \quad \phi(z_1) = \frac{1}{(e^{i\pi/3}+1)(i\sqrt{3})} = \frac{1}{\left(\frac{3}{2} + i\frac{\sqrt{3}}{2}\right)(i\sqrt{3})} = \frac{\frac{3}{2} - i\frac{\sqrt{3}}{2}}{(3)(i\sqrt{3})}$$

So then

$$\operatorname*{Res}_{z=z_1} \frac{1}{z^3+1} = \frac{\frac{3}{2} - i\frac{\sqrt{3}}{2}}{3i\sqrt{3}}$$

and

$$\begin{aligned} (1 - e^{i2\pi/3}) \int_0^\infty \frac{dr}{r^3+1} &= 2\pi i \left(\frac{\frac{3}{2} - i\frac{\sqrt{3}}{2}}{3i\sqrt{3}} \right) \\ \left(\frac{3}{2} - i\frac{\sqrt{3}}{2}\right) \int_0^\infty \frac{dr}{r^3+1} &= \frac{2\pi}{3\sqrt{3}} \left(\frac{3}{2} - i\frac{\sqrt{3}}{2}\right). \end{aligned}$$

Cancel the terms in parentheses.

$$\int_0^\infty \frac{dr}{r^3+1} = \frac{2\pi}{3\sqrt{3}}$$

Therefore, changing the dummy integration variable to x,

$$\boxed{\int_0^\infty \frac{dx}{x^3+1} = \frac{2\pi}{3\sqrt{3}}.}$$

The Integral Over C_R

Our aim here is to show that the integral over C_R tends to zero in the limit as $R \to \infty$. The parameterization of the circular arc in Fig. 93 is $z = Re^{i\theta}$, where θ goes from 0 to $2\pi/3$.

$$\begin{aligned}\int_{C_R} \frac{dz}{z^3+1} &= \int_0^{2\pi/3} \frac{Rie^{i\theta}\,d\theta}{(Re^{i\theta})^3+1} \\ &= \int_0^{2\pi/3} \frac{Rie^{i\theta}\,d\theta}{R^3e^{i3\theta}+1}\end{aligned}$$

Now consider the integral's magnitude.

$$\begin{aligned}\left|\int_{C_R} \frac{dz}{z^3+1}\right| &= \left|\int_0^{2\pi/3} \frac{Rie^{i\theta}\,d\theta}{R^3e^{i3\theta}+1}\right| \\ &\le \int_0^{2\pi/3} \left|\frac{Rie^{i\theta}}{R^3e^{i3\theta}+1}\right| d\theta \\ &= \int_0^{2\pi/3} \frac{\left|Rie^{i\theta}\right|}{|R^3e^{i3\theta}+1|}\, d\theta \\ &= \int_0^{2\pi/3} \frac{R}{|R^3e^{i3\theta}+1|}\, d\theta \\ &\le \int_0^{2\pi/3} \frac{R}{|R^3e^{i3\theta}|-|1|}\, d\theta \\ &= \int_0^{2\pi/3} \frac{R}{R^3-1}\, d\theta \\ &= \frac{2\pi}{3}\frac{R}{R^3-1}\end{aligned}$$

Now take the limit of both sides as $R \to \infty$.

$$\begin{aligned}\lim_{R\to\infty}\left|\int_{C_R} \frac{dz}{z^3+1}\right| &\le \lim_{R\to\infty} \frac{2\pi}{3}\frac{R}{R^3-1} \\ &= \lim_{R\to\infty} \frac{2\pi}{3R^2}\frac{1}{1-\frac{1}{R^3}}\end{aligned}$$

The limit on the right side is zero.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{dz}{z^3+1}\right| \le 0$$

The magnitude of a number cannot be negative.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{dz}{z^3+1}\right| = 0$$

The only number that has a magnitude of zero is zero. Therefore,

$$\lim_{R\to\infty}\int_{C_R} \frac{dz}{z^3+1} = 0.$$